growth can be stimulated; second, electrical techniques such as this one may be useful for study of droplet nucleation and growth; and third, the successful operation of the high-temperature loop is an indication that alkali-metal systems may be operable at somewhat higher temperatures than are now proposed.

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Stability of Damped Mechanical Systems

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THE stability of damped, mechanical systems is of great interest in space dynamics. The main observation of this note is that for some systems the Hamiltonian function of mechanics is a very useful "testing function" for Lyapunov stability. The Hamiltonian differs from the total energy in the important case of gyroscopic systems. Certain statements about mechanics will be made, and then theorems on mechanical stability will be stated.

Hamiltonian Systems

If the equations of motion of a mechanical system are written in Hamiltonian form, the result is [1]

$$\dot{p}_i = -(\partial H/\partial q_i) + Q_i \qquad (i = 1,2,\ldots N)$$
$$\dot{q}_i = (\partial H/\partial p_i) \tag{1}$$

where $H(p,q)$ is the Hamiltonian function, assumed to be free of explicit dependence on time, the q_i are generalized coordinates of the problem, and the p_i are generalized momenta. Q_i is a generalized force, not derived from a potential function. The equations of motion (1) can be seen to imply the power balance relation

$$\dot{H} = P = \sum_{i=1}^{N} Q_i\,\dot{q}_i \tag{2}$$

where $P = \dot{H}$ is called the power. If $\dot{H}$ is calculated from the kinetic and potential energy expressions, it is

$$H = T_2 + U \tag{3}$$

where $T = T_2 + T_1 + T_0$, T_n is a homogeneous form of nth degree in the $\dot{q}_i$, and $U = V - T_0$. The proof of (3) is immediate from Euler's theorem on nth degree homogeneous forms.

It is important to observe that, if the total energy is defined as $E = T + V$, then

$$E - H = T_1 + 2T_0$$

This expression shows that there is a difference between E and H that is, in general, dependent on time.

Stability Theorems

Based upon the stability theory of Lyapunov[2,3] the definition of stability may be stated. It is assumed that the equilibrium point $[(\partial H/\partial p_i) = 0,\ (\partial H/\partial q_i) = 0]$ is $\mathbf{x} = 0$ where $\mathbf{x}(t) = (q_1,q_2, \ldots q_N,p_1,p_2 \ldots p_N)$. If, given a system of differential equations (1), there exists an initial phase vector $\mathbf{x}(0)$ small enough $[\|\mathbf{x}(0)\| \leq \epsilon > 0]$ so that $\|\mathbf{x}(t)\|$ is within an arbitrary bound $[\|\mathbf{x}(t)\| \leq \delta > 0]$, then (1) is said to be stable. If, in addition to being stable, $\|\mathbf{x}\| \to 0$ as $t \to \infty$, then (1) is said to be asymptotically stable. Based upon this definition of stability (a more precise definition is stated in the references), we state the following important result.

Theorem

If for the autonomous mechanical systems described by (1) the power $P = \dot{H}$ is negative definite in a region S of the $\mathbf{x}$-space, then the motions are: 1) asymptotically stable if $H(p,q)$ is positive definite in S, or 2) unstable if $H(p,q)$ is sign variable or negative definite in S.

Part 1 of the theorem is proved using Lyapunov's theorem on asymptotic stability with H as a "testing function." Part 2 is proved using Lyapunov's theorem on instability with H as a "testing function."[4]

A corollary to the preceding theorem may be stated after observing that a necessary and sufficient condition for asymptotic stability is that H be positive definite in $\mathbf{x}$ if P is negative definite. Since (3) shows H to be the sum of T_2, a positive definite quadratic form in $\dot{q}_i$, and U to be a function of q_i, we see that U must be a positive definite function of q_i in order for H to be positive definite.

Corollary 1

If the hypothesis of the theorem is satisfied, then the motions are: 1) asymptotically stable if $U(q)$ is positive definite, or 2) unstable if $U(q)$ is nonpositive definite in the q_i.

Corollary 2

Corollary 2 is an important qualitative result. If the system obeys the hypothesis of the theorem, then its stability behavior cannot depend upon the magnitude or the analytical form of the power function. This result is true because the function U does not contain parameters associated with the sign of P. Testing for stability reduces to examining only U if P is negative. A very important corollary to the theorem is proved by Lefschetz and La Salle.[3] This corollary states that the theorem is true even if P is only negative semidefinite as long as there are no "decoupled motions."

Corollary 3

The condition of the theorem that P must be negative definite in $\mathbf{x}$ may be replaced by 1) $\dot{H} = P \leqq 0$ for all $\mathbf{x}$, and 2) $\dot{H} = P$ does not vanish identically (for all t) for motion not at the point $\mathbf{x} = 0$.

Results

The results of this note have been found to be important and useful in treating various space dynamics problems. It is imperative that we keep the fundamental distinction between E and H in mind in space problems involving rotating coordinates or cyclic variables. The results of this note with an expanded treatment and many applications will be found in Ref. 4.

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